Centre for Theoretical Physics
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TENTAMEN GENERAL RELATIVITY

friday, 4-02-2005, room 5116.0116, 14.00-17.00

Indicate at the first page clearly your name, address, date of birth, year of arrival and at every other page your name.

Question 1

Consider a Riemannian manifold with a covariantly constant metric $g_{ab}(x)$ and Christoffel symbols $\Gamma^c_{ab}(x) = \Gamma^c_{ba}(x)$.

(1.1) Use the fact that the metric is covariantly constant, i.e.

$$\nabla_c g_{ab}(x) = 0\,, \tag{1}$$

to solve for the Christoffel symbols in terms of the metric.

(1.2) Consider a curve $x^a(s)$ with affine parameter s. The geodesic equation for this curve is given by

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0\,, \tag{2}$$

where the dot indicates differentiation with respect to s. Show that the geodesic equation (2) follows from the Euler-Lagrange equations corresponding to the Lagrangian

$$L = \tfrac{1}{2}\dot{x}^a\dot{x}^b g_{ab}(x)\,. \tag{3}$$

(1.3) Consider the metric ($c = 1$)

$$\begin{aligned} ds^2 &= dt^2 - dz^2 - L^2(e^{2\beta}dx^2 + e^{-2\beta}dy^2) \\ &= dudv - L^2(e^{2\beta}dx^2 + e^{-2\beta}dy^2)\,, \end{aligned} \tag{4}$$

with $u = t - z, v = t + z$ and $L = L(u), \beta = \beta(u)$ arbitrary functions of u. This metric describes an exact plane gravitational wave solution to the Einstein equations. Calculate the Christoffel symbols corresponding to the metric (4) in (u, v, x, y) coordinates.

(1.4) Show that the curves

$$\dot{t} = 1 \,, \quad \dot{x} = \dot{y} = \dot{z} = 0 \tag{5}$$

are timelike geodesics.

Question 2

Consider a free test particle (massive or massless) moving along a geodesic curve in the Schwarzschild metric ($c = 1$)

$$ds^2 = (1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \,, \tag{6}$$

which for $r > 2m$ describes the space-time outside a black hole. Using spherical symmetry we can restrict ourselves to the equatorial plane $\theta = \frac{\pi}{2}$.

(2.1) The metric (6) is independent of the coordinates t and ϕ. Use this to show that the quantities

$$\mathcal{E} = g_{t\mu}\dot{x}^\mu \quad \text{and} \quad \mathcal{L} = g_{\phi\mu}\dot{x}^\mu \tag{7}$$

are constants of motion. Note: $\dot{x}^\mu$ in (7) represents the derivative of x^μ with respect to the proper time τ of the geodesic curve $x^\mu(\tau)$. What is the physical interpretation of $\mathcal{E}$ and $\mathcal{L}$?

(2.2) Proof the following relation

$$\dot{r}^2 = \mathcal{E}^2 - \left(1 - \frac{2m}{r}\right)\left(\delta + \frac{\mathcal{L}^2}{r^2}\right), \tag{8}$$

where

$$\delta = \begin{cases} 1 & \text{for massive particles} \,, \\ 0 & \text{for massless particles} \,. \end{cases} \tag{9}$$

The relation (5) resembles the energy relation for a particle with mass 2 and energy $\mathcal{E}^2$ that moves in a potential

$$V(r) = \left(1 - \frac{2m}{r}\right)\left(\delta + \frac{\mathcal{L}^2}{r^2}\right). \tag{10}$$

(2.3) Make a graph of the potential $V(r)$ for massless particles and determine for which value of r the potential is maximum. For which value of r is circular motion of light possible? Is this circular orbit stable?

(2.4) Consider the potential for massive particles. For which values of $\mathcal{L} > 0$ has the potential $V(r)$ no extrema? For larger values of $\mathcal{L}$ the potential $V(r)$ has a maximum and a minimum. Make a graph of the potential $V(r)$. Show that only for $r > 6m$ stable circular orbits for massive particles are possible.

Question 3

The Robertson-Walker metric for $k = 1$ can be written in the form (we take $c = 1$)

$$ds^2 = dt^2 - R(t)^2\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\}. \tag{11}$$

(3.1) Show that for geodesics with $\dot{\theta} = \dot{\phi} = 0$ the quantity

$$R(t)^2\dot{\chi} \tag{12}$$

is constant. The dot indicates differentiation with respect to an affine parameter.

(3.2) Show that for lightlike geodesics with $\dot{\theta} = \dot{\phi} = 0$

$$R(t)\frac{d\chi}{dt} = \pm 1. \tag{13}$$

We consider a closed Friedmann universe. The function $R(t)$ corresponding to such a universe satisfies the differential equation

$$\left(\frac{dR}{dt}\right)^2 + 1 = \frac{A^2}{R}, \tag{14}$$

where A is a constant. We impose the boundary condition that $R = 0$ for $t = 0$.

(3.3) Show that the solution of the differential equation (14) is given by the equations

$$\begin{aligned} R &= \tfrac{1}{2}A^2(1 - \cos\psi)\,, \\ t &= \tfrac{1}{2}A^2(\psi - \sin\psi)\,, \end{aligned} \tag{15}$$

where ψ is a parameter. Give the graph of the function $R(t)$.

(3.4) At a time $t_1 << A^2$ a foton is emitted from a point P and this foton starts following a geodesic in the plane with $\theta = \phi = \pi/2$. The point P has constant spacelike coordinates $\chi = 0$ and $\theta = \phi = \pi/2$. Calculate the time it takes for the foton to return to P.